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Perturbed harmonic oscillator ladder operators: eigenenergies and eigenfunctions for the $X^2+\lambda X^2/(1+gX^2)$ interaction

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Abstract. The perturbed ladder operators method is applied to the resolution of the perturbed harmonic oscillator wave equation for the case where the perturbation is expandable in a convergent series of Hermite polynomials H_k, i.e. when

$$\mathscr{V}(X)=b^2X^2+\sum_k C_kH_k(b^{1/2}X).$$

It is found that the use of an Hermite polynomials basis, together with the use of binomial coefficient functions in the quantum number, greatly simplifies the determination of the perturbed ladder and factorisation functions. Thus, one obtains analytical expressions of the eigenenergies and eigenfunctions up to any order of the perturbation, without increasing intricacy. Thorough calculation has been given for a perturbing potential $\mathscr{V}(X)$ function even in X. As an illustrative application of the procedure, the resolution of the Schrödinger equation with a potential function $\mathscr{V}(X)=X^2+\lambda X^2/(1+gX^2)$, $g>0$ is reinvestigated.

1. Introduction

In the present paper, the perturbed ladder operators method (Infeld and Hull 1951, Bessis *et al* 1978, 1980, 1981, to be referred to as I, II and III, respectively) is reformulated in order to be efficiently applied to the resolution of eigenequations which can be viewed as perturbed harmonic oscillator equations. The resolution of the one-dimensional Schrödinger equation

$$(\mathrm{d}^2/\mathrm{d}X^2-\mathscr{V}(X)+\mathscr{E})\psi(X)=0 \qquad -\infty<X<\infty \tag{1}$$

with the following interaction potential

$$\mathscr{V}(X)=X^2+\lambda X^2/(1+gX^2) \qquad g>0 \tag{2}$$

will constitute the motivating example of this necessary reformulation.

As pointed out elsewhere this type of interaction is of interest in several areas of physics which have been summarised by several authors (see, for instance, Mitra 1978 and Kaushal 1979). In particular, this type of potential occurs when considering models in laser theory (Risken and Vollmer 1967, Haken 1970), and in quantum field theory with a non-linear Lagrangian (Biswas *et al* 1973). Moreover, for particular negative values of λ, this two-parameter potential could be used as a model of a double

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well potential. On the other hand, from a computational point of view, this type of potential is a particular illustration of the difficulties encountered when solving the problem by the straightforward variational or perturbational procedures (Mitra 1978, Kaushal 1979, Hautot 1981). More efficient variational treatments, either using a refined non-perturbative Hill method (Hautot 1981) or using properly scaled (in λ and g) harmonic oscillator functions as a basis set (Bessis and Bessis 1980), have been applied. Results obtained are, on the whole, in accordance with those given by Mitra. Recently, Flessas and others (Flessas 1981, Varma 1981, Whitehead *et al* 1982, Lai and Lin 1982) have also shown the existence of a class of exact solutions for particular values of λ and g.

Since $\mathscr{V}(X)$ is readily expandable in powers of X^2, one could naively consider that the solution of the eigenequation (1) is straightforward either by the classical Rayleigh–Schrödinger perturbation scheme or by the perturbed ladder operators scheme. Nevertheless, the perturbation series does not converge for any values of λ and g. In the present paper, it is shown that this difficulty can be overcome as long as the potential function can be expanded in a convergent series on the basis of the Hermite polynomials. Therefore, the eigenequation (1) is considered as a perturbed harmonic oscillator type wave equation and the perturbed ladder operators method is applied to the solution of the eigenequation

$$\left(\frac{d^2}{dX^2}-b^2X^2-\sum_k C_{2k}H_{2k}(b^{1/2}X)+\mathscr{E}\right)\psi(X)=0 \tag{3}$$

where the 'scaling real parameter' b is introduced in order to improve the zeroth-order harmonic Hamiltonian (Bessis and Bessis 1980).

The principles of the perturbed ladder operators method are briefly recalled in §2. In §3, an improved formulation of the method is presented which leads to compact analytical expressions of the ladder operators, factorisation function, eigenvalues and eigenfunctions of equation (3) in terms of the quantum numbers. Indeed, many simplifications occur when introducing an 'Hermite polynomial basis' as the working basis instead of the usual X^n basis. Then, finding the λ- and g-dependent eigenvalues and eigenfunctions for the potential function $\mathscr{V}(X)$, up to the Nth order of the perturbation, is a simple matter of straightforward application of the preceding formulae: the only preliminary step is to compute the expansion coefficients of $\mathscr{V}(X)$ on Hermite polynomials (§4). Since extensive numerical results have already been published for a large range of values of λ and g (Hautot 1981, Bessis and Bessis 1980 and references therein), we found it unnecessary to give numerical results again. In §§3 and 4, our attention has been focused on the case of the harmonic oscillator perturbed by a potential function even in X. In the last section (§5), the formulation is extended to the case of a perturbing potential function without any defined parity in X.

2. The perturbed ladder operators method

Let us consider a second-order differential eigenequation which has been first reduced to the standard form

$$(d^2/dx^2-U(x,m)+\Lambda_{jm})\psi_{jm}=0 \tag{4}$$

associated with the boundary conditions ($x_1 \leqslant x \leqslant x_2$)

$$|\psi(x_1)|^2 = |\psi(x_2)|^2 = 0 \qquad \int_{x_1}^{x_2} |\psi(x)|^2 \, dx = 1 \tag{5}$$

where m and j are quantum numbers which take successive discrete values labelling the eigenvalues and eigenfunctions.

In previous papers (see I–III), it has been shown how, by starting from an unperturbed problem with a potential function $U^{(0)}(x, m)$ leading to a factorisable equation, one can build up perturbed ladder and factorisation functions $K(x, m)$ and $L(m)$ allowing the factorisation of eigenequation (4), up to a given order N of the perturbation. Writing the potential, the ladder and the factorisation functions in series of the perturbation parameter η, one obtains

$$\begin{aligned} U(x, m) &= U^{(0)}(x, m) + \eta U^{(1)}(x, m) + \eta^2 U^{(2)}(x, m) + \ldots + \eta^N U^{(N)}(x, m) \\ K(x, m) &= K^{(0)}(x, m) + \eta K^{(1)}(x, m) + \eta^2 K^{(2)}(x, m) + \ldots + \eta^N K^{(N)}(x, m) \\ L(m) &= L^{(0)}(m) + \eta L^{(1)}(m) + \eta^2 L^{(2)}(m) + \ldots + \eta^N L^{(N)}(m) \end{aligned} \tag{6}$$

where $U^{(0)}$ is one of the six Infeld–Hull exact factorisation types and $K^{(0)}$ and $L^{(0)}$ are the associated ladder and factorisation functions allowing the factorisation of equation (4) with $U^{(0)}$.

As has been already outlined in paper III, the necessary and sufficient condition to be fulfilled by the required functions is, at each order N of the perturbation,

$$\begin{aligned} &\Delta\{[\mathrm{d}/\mathrm{d}x - 2K^{(0)}(x, m)]K^{(N)}(x, m)\} - 2(\mathrm{d}/\mathrm{d}x)K^{(N)}(x, m+1) \\ &\qquad = \Delta\left(L^{(N)}(m) + \sum_{\nu=1}^{N-1} K^{(\nu)}(x, m)K^{(N-\nu)}(x, m)\right) \end{aligned} \tag{7a}$$

$$\begin{aligned} &[\mathrm{d}/\mathrm{d}x - 2K^{(0)}(x, m)]K^{(N)}(x, m) \\ &\qquad = -U^{(N)}(x, m) + L^{(N)}(m) + \sum_{\nu=1}^{N-1} K^{(\nu)}(x, m)K^{(N-\nu)}(x, m) \end{aligned} \tag{7b}$$

where Δ is the usual first difference operator in m so that

$$\Delta F(m) = F(m+1) - F(m). \tag{8}$$

Of course, specific expressions for the $U^{(\nu)}$, $K^{(\nu)}$ and $L^{(\nu)}$ correspond to each exact Infeld–Hull factorisation type. The finite-difference aspect in m of equation (7a) determines the m-dependence of the functions while its differential aspect in x determines their x-dependence. These equations are solved recursively: i.e. when determining $K^{(N)}$ and $L^{(N)}$, it is assumed that all the $K^{(\nu)}$ and $L^{(\nu)}$, for $\nu = 1, 2, \ldots, N-1$, have already been found. The first equation (7a) is used to determine the ladder and factorisation functions $K^{(N)}(x, m)$ and $L^{(N)}(m)$. Once they are known the potential functions $U^{(N)}(x, m)$ are given by (7b) and one obtains the required 'factorising' potential function $U(x, m)$ of the eigenequation (4).

Thus, one can solve physico-mathematical problems with a potential function $V(x, m)$ such as

$$V(x, m) = V^{(0)}(x, m) + \eta V^{(1)}(x) + \eta^2 V^{(2)}(x) + \ldots + \eta^N V^{(N)}(x) \tag{9}$$

where $V^{(0)}(x, m) = U^{(0)}(x, m)$ and the $V^{(\nu)}(x)$ have the same dependence in x as the $U^{(\nu)}(x, m)$. In order to match $V(x, m)$ with the factorising potential $U(x, m)$, one has to resort to the 'artificial' factorisation process. One considers the $U(x, m)$ potential as 'embedded' in a potential function $u(x, m, \mu)$ which depends on a supplementary 'artificial' parameter μ such that $u(x, m, \mu)$ can be identified in m with $V(x, m)$ and that $u(x, m, \mu = m) = U(x, m)$. Then the following condition must hold, for any x,

$$V^{(\nu)}(x) = U^{(\nu)}(x, m = \mu). \tag{10}$$

Finally, one can factorise an eigenequation (4) with a given potential function $V(x, m)$ by determining the associated perturbed ladder and factorisation functions which are solutions of the difference–differential equation (7a) and satisfy the following condition

$$(\mathrm{d}/\mathrm{d}x - 2K^{(0)}(x, \mu))K^{(N)}(x, \mu) = -V^{(N)}(x) + L^{(N)}(\mu) + \sum_{\nu=1}^{N-1} K^{(\nu)}(x, \mu)K^{(N-\nu)}(x, \mu). \tag{11}$$

Thus, the perturbed ladder and factorisation functions $K^{(\nu)}(x, m; \mu)$ and $L^{(\nu)}(m; \mu)$ associated with the 'embedding' potential $u(x, m, \mu)$ both depend on the parameter μ. Once they are known, the perturbed problem (up to the Nth order) may be handled in the same way as the exact factorisable (unperturbed) problem. The necessary condition for the existence of quadratically integrable wavefunctions, i.e. the quantisation condition, is $\varepsilon(j - |m|) = v = \text{integer} \geqslant 0$ where $\varepsilon = 1$ (or $\varepsilon = -1$) according to whether the zeroth-order factorisation function $L^{(0)}(m)$ is an increasing (or decreasing) function of m. The required total eigenvalues are obtained from $L(m)$ by setting $m = j + \frac{1}{2} + \frac{1}{2}\varepsilon$ and $\mu = m$ in the expression of $L^{(\nu)}(m, \mu)$

$$\Lambda_{jm} = L^{(0)}(m = j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) + \sum_{\nu=1}^{N} \eta^{\nu} L^{(\nu)}(m = j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon; \mu = m). \tag{12}$$

The required eigenfunctions with the given potential $V(x, m)$ can be obtained from the knowledge of the perturbed 'key function' ψ_{jj} either stepwise by the use of the ladder operators $H_m^{\pm} = K(x, m; \mu) \mp \mathrm{d}/\mathrm{d}x$ or by the use of a three-term recursion relation (see paper II). They can also be obtained as a linear combination of the unperturbed functions (see paper III). In the present paper an alternative and more direct procedure is proposed.

3. Eigenvalues and eigenfunctions of the perturbed harmonic oscillator

The wave equation (3) for an harmonic oscillator perturbed by a potential function, even in X, can be conveniently written

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}X^2} - b^2X^2 + b(2m+1) - b\sum_{\nu=1}^{N} V^{(\nu)}(b^{1/2}X) + b\Lambda_{jm}\right)\psi_{jm} = 0 \tag{13}$$

with $-\infty < X < \infty$ and

$$V^{(\nu)}(b^{1/2}X) = \sum_{k=1}^{S_\nu+1} \frac{k!}{(2k)!} d_{2k}^{(\nu)} H_{2k}(b^{1/2}X). \tag{14}$$

The $H_{2k}(x)$ are Hermite polynomials and the factor $k!/(2k)! = (-)^k (H_{2k}(0))^{-1}$ has been introduced for computational convenience. Concerning the actual choice of the set of upper bounds S_ν, see equation (21) and the associated commentary.

For ($N=0$)† the wave equation (13) reduces to the factorisable (Infeld–Hull type D) wave equation with

$$x=b^{1/2}X \qquad U^{(0)}(x,m)=x^2-(2m+1)$$
$$K^{(0)}(x,m)=x \qquad L^{(0)}(m)=-2m \qquad m-j=v=\text{positive integer.} \tag{15}$$

Therefore, the harmonic oscillator eigenvalue is

$$E_v^{(0)}=b(2m+1)+b\Lambda_j^{(0)}=2b(v+1/2). \tag{16}$$

The normalised harmonic oscillator eigenfunctions are

$$\psi_v^{(0)}(X)=(b/\pi)^{1/4}(1/2^v v!)^{1/2}\exp(-bX^2/2)H_v(b^{1/2}X). \tag{17}$$

3.1. *m-dependence of the perturbed ladder and factorisation functions $K^{(N)}(x,m)$ and $L^{(N)}(m)$*

Since the perturbed potential function of the eigenequation (13) is even in x, for the sake of simplicity we apply the perturbation scheme with a factorising potential function $U(x,m)$ even in x. The general case is examined in §5. Following from the factorisability conditions (7) and from parity considerations, it can be directly inferred (see paper III) that the perturbed ladder function is odd in x and one can set

$$K^{(N)}(x,m)=\sum_{t=0}^{S_N}\gamma_{2t+1}^{(N)}(m)H_{2t+1}(x) \tag{18}$$

where the $\gamma_{2t+1}^{(N)}(m)$ functions have to be determined in order to satisfy condition ($7a$).

When using the above expression of $K^{(N)}$ and expression (15) of $K^{(0)}$, it follows that both sides of the factorisability condition ($7a$) are x-polynomials of even parity and, thus, can be expanded on the basis of the Hermite polynomials ($H_{2t}(x)$). The familiar properties of the Hermite polynomials can be used (see equation (A1.4) of appendix 1) and when equating the coefficients of $H_{2t}(x)$ on both sides of equation ($7a$), one gets for $t=0$ to S_N, the finite difference equation to be satisfied by the required $\gamma_{2t+1}^{(N)}$ functions

$$\Delta\gamma_{2t+1}^{(N)}(m)=-4(2t+3)\gamma_{2t+3}^{(N)}(m)-\Delta w_{2t+2}^{(N)}(m). \tag{19}$$

The $w_{2t+2}^{(N)}(m)$ functions originate from the preceding orders of the perturbation and are defined by

$$\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)=\sum_{t=0}^{S_N+1}w_{2t}^{(N)}(m)H_{2t}(x). \tag{20}$$

Details of their calculation is given in appendix 2.

It should be noted that, while at the first order $N=1$ of the perturbation, the upper bound S_1 which is involved in $K^{(1)}(x,m)$ can be chosen arbitrarily, this is not true for the higher S_N. Indeed, owing to the presence of the term (20) in the factorisability condition (7), it is found that the following condition must be fulfilled (see appendix 2)

$$S_N=NS_1. \tag{21}$$

† Indeed $\Sigma_{\nu=1}^{0}=0$ since its algorithmic signification corresponds to the neutral operation for the addition. Thus the equation (13) is also valid for $N=0$.

The finite difference equation (19) can be solved recursively, the integer t descending stepwise from $t = S_N$ down to zero. It is easily inferred that both $\gamma_{2t+1}^{(N)}(m)$ and $w_{2t+2}^{(N)}$ are polynomials in m of degree $(S_N - t)$ (see appendix 2). From the factorisability condition (7a), it follows that $L^{(N)}(m)$ is a polynomial in m of degree $(S_N + 1)$.

As will be seen afterwards, it is rewarding to use Newton's backward formula (Jordan 1965) and to set

$$\gamma_{2t+1}^{(N)}(m) = \sum_{u=0}^{S_N - t} \binom{m-\mu+u-1}{u} [\Delta^u \gamma_{2t+1}^{(N)}(m)]_{m=\mu-u} \tag{22}$$

where μ is the 'artificial' parameter of the 'embedded factorisation' procedure and the $\binom{m}{k}$ are binomial functions. This expansion is to be preferred to power expansion in m, since the binomial functions satisfy the simple difference equation $\Delta\binom{m}{k} = \binom{m}{k-1}$.

Finally, following from the above considerations, the total ladder and factorisation functions can be written

$$\begin{aligned} K(x,m) &= x + \sum_{\nu=1}^{N} \eta^\nu \sum_{u=0}^{S_\nu} \binom{m-\mu+u-1}{u} \sum_{t=0}^{S_\nu - u} A_{2t+1}^{(\nu)}(u) H_{2t+1}(x) \\ L(m) &= -2m + \sum_{\nu=1}^{N} \eta^\nu \sum_{u=0}^{S_\nu+1} \binom{m-\mu+u-1}{u} \alpha_u^{(\nu)} \end{aligned} \tag{23}$$

where the coefficients $A_{2t+1}^{(\nu)}(u) = [\Delta^u \gamma_{2t+1}^{(\nu)}(m)]_{m=\mu-u}$ and $\alpha_u^{(\nu)}$ have to be found.

At this stage of the calculation, rather than determining the coefficients $A_{2t+1}^{(\nu)}(u)$ by solving the finite difference equation (19), it is more convenient to take advantage of the knowledge of the m-dependence of the $K^{(\nu)}(x, m)$ and to solve in x the factorisability condition (equations (7a) and (11)).

3.2. x-dependence of the perturbed ladder function $K^{(N)}(x, m)$

Let us set

$$\begin{aligned} K^{(N)}(x,m) &= \sum_{u=0}^{S_N} \binom{m-\mu+u-1}{u} P_u^{(N)}(x) \\ \sum_{\nu=1}^{N-1} K^{(\nu)}(x,m) K^{(N-\nu)}(x,m) &= \sum_{u=0}^{S_N} \binom{m-\mu+u-1}{u} Q_u^{(N)}(x) \end{aligned} \tag{24}$$

where

$$P_u^{(N)}(x) = \sum_{t=0}^{S_N - u} A_{2t+1}^{(N)}(u) H_{2t+1}(x) \tag{25}$$

and (see appendix 2)

$$Q_u^{(N)}(x) = \sum_{t=0}^{S_N - u + 1} B_{2t}^{(N)}(u) H_{2t}(x). \tag{26}$$

From the factorisability condition (11), one gets

$$(\mathrm{d}/\mathrm{d}x - 2x) P_0^{(N)}(x) = -V^{(N)}(x) + \alpha_0^{(N)} + Q_0^{(N)}(x). \tag{27}$$

After substituting for $K^{(N)}$, $\sum_{\nu=1}^{N-1} K^{(\nu)} K^{(N-\nu)}$ and $L^{(N)}$ from equations (24) and (23) into the factorisability condition (7a), and equating the coefficients of the binomials $\binom{m-\mu+u}{u}$ on both sides of (7a) one finds, for $u = 0$ to $S_N - 1$,

$$(\mathrm{d}/\mathrm{d}x - 2x) P_{u+1}^{(N)}(x) = 2(\mathrm{d}/\mathrm{d}x) P_u^{(N)}(x) + \alpha_{u+1}^{(N)} + Q_{u+1}^{(N)}(x). \tag{28}$$

Owing to the raising property of the operator $(\mathrm{d}/\mathrm{d}x - 2x)$ when acting on Hermite polynomials (see appendix 1), the left sides of the expressions (27) and (28) do not involve the Hermite polynomial $H_0(x)$. Consequently

$$\begin{aligned}\alpha_0^{(N)} &= -[Q_0^{(N)}(x)]_{H_0}\\ \alpha_{u+1}^{(N)} &= -[2(\mathrm{d}/\mathrm{d}x)P_u^{(N)}(x) + Q_{u+1}^{(N)}(x)]_{H_0}\end{aligned}\tag{29}$$

where the symbol $[\;]_{H_0}$ denotes the projection of the inner part of the bracket onto the Hermite polynomial H_0.

When introducing the right inverse integral operator $\hat{D}$ of $(\mathrm{d}/\mathrm{d}x - 2x)$ (see equations (A1.5) and (A1.6)) one gets

$$P_0^{(N)}(x) = \hat{D}[Q_0^{(N)}(x) - V^{(N)}(x)]\tag{30}$$

$$P_{u+1}^{(N)}(x) = \hat{D}[2(\mathrm{d}/\mathrm{d}x)P_u^{(N)}(x) + Q_{u+1}^{(N)}(x)]\tag{31}$$

where $Q_0^{(N)}(x)$ and the $Q_{u+1}^{(N)}(x)$ are known in terms of the $P_u^{(\nu)}(x)$ generated in the preceding orders of the perturbation (see appendix 2). Now, applying successively the relation (30) for $N = 1$, $N = 2, \ldots,$ one directly obtains the $P_0^{(1)}(x)$, $P_0^{(2)}(x), \ldots$ and $P_0^{(N)}(x)$ functions in terms of the perturbing parts $V^{(\nu)}(x)$ of the potential function $V(x)$ which contains the parameters of the physico-mathematical problem under consideration. The repeated use of the relation (31) yields the expression of the $P_{u+1}^{(N)}(x)$ in terms of the $P_0^{(\nu)}(x)$, $\nu = 1$ to N. One finds

$$P_u^{(N)}(x) = (2\hat{D}\ \mathrm{d}/\mathrm{d}x)^u P_0^{(N)}(x) + \sum_{t=0}^{u-1} (2\hat{D}\ \mathrm{d}/\mathrm{d}x)^t \hat{D} Q_{u-t}^{(N)}(x).\tag{32}$$

The expansion of $P_u^{(N)}(x)$ on the Hermite polynomials basis, i.e. the expressions of the coefficients $A_{2t+1}^{(N)}(u)$ in equation (25), are obtained from the operator relations (30), (32) and (A1.7). One finds

$$\begin{aligned}A_{2t+1}^{(N)}(u) = (-)^u (2^{t+1-u}(2t+1)!!)^{-1} d_{2t+2u+2}^{(N)}\\ - \sum_{\sigma=0}^{u} (-4)^{u-\sigma} \frac{(2t+2u-2\sigma+1)!!}{(2t+1)!!} B_{2t+2u+2-2\sigma}^{(N)}(\sigma)\end{aligned}\tag{33}$$

where $u = 0$ to S_N and $t = 0$ to $S_N - u$.

The $d_{2k}^{(N)}$ are the coefficients of the perturbing part $V^{(N)}(x)$ of the 'physical' potential (14) and the $B_{2t}^{(N)}(u)$ coefficients are known in terms of the $A_{2t+1}^{(\nu)}(u)$ coefficients of the preceding orders of the perturbation (see appendix 2)

$$\begin{aligned}B_{2t}^{(N)}(u) = \sum_{s=0}^{u} \sum_{\sigma=0}^{u} (-)^{u+\sigma+s} \binom{u}{s} \binom{s}{u-\sigma} \sum_{\nu=1}^{N-1} \sum_{k=0}^{S_\nu - s} \sum_{q=0}^{S_{N-\nu}-\sigma}\\ \times h(2k+1, 2q+1; 2t) A_{2k+1}^{(\nu)}(s) A_{2q+1}^{(N-\nu)}(\sigma)\end{aligned}\tag{34}$$

where $u = 0$ to S_N and $t = 0$ to $(S_N - u + 1)$. The coefficients $h(\;)$ originate from the expansion of the product of two Hermite polynomials as a linear combination of Hermite polynomials (see equation (A1.3)).

For practical computation, it is convenient to include the whole perturbation of the potential in $V^{(1)}$ and write $V(x, m) = U^{(0)}(x, m) + V^{(1)}(x, m)$ i.e. $d_{2k}^{(1)} \equiv d_{2k}$ and $d_{2k}^{(\nu)} \equiv 0$ for $\nu \geq 2$. For $N = 1$, since the $B_{2t}^{(1)}(u) \equiv 0$, the expression (33) reduces to

$$A_{2t+1}^{(1)}(u) = (-)^u [2^{t+1-u}(2t+1)!!]^{-1} d_{2t+2u+2}.\tag{35}$$

For $N \geqslant 1$, one obtains

$$A_{2t+1}^{(N)}(u)=\sum_{l=0}^{u}(-)^{u-l+1}\frac{2^{2u-2l}(2t+2u-2l+1)!!}{(2t+1)!!}B_{2t+2u-2l+2}^{(N)}(l). \tag{36}$$

Finally, the $B^{(N)}$ and $A^{(N)}$ are determined recursively in terms of the d_{2k}, via the $B^{(\nu)}$ and $A^{(\nu)}$ ($\nu=2$ to $N-1$) already found, by means of relations (34) and (36) and the analytical expression of the total ladder function follows from (23).

3.3. *Eigenvalues of the perturbed harmonic oscillator*

The eigenvalue $b\Lambda_{jm}$ of the eigenequation (13) is directly obtained from equation (12) and expression (23) of $L(m)$. After setting $m=j$, $\mu=m$ and introducing the quantisation condition $m-j=v$ in the expression of $L(m)$, the perturbed harmonic oscillator eigenvalue is obtained

$$E_v=2b(v+\tfrac{1}{2})+\sum_{\nu=1}^{N}\eta^{\nu}E_v^{(\nu)} \tag{37}$$

where

$$E_v^{(\nu)}=b\sum_{u=0}^{S_\nu+1}(-)^u\binom{v}{u}\alpha_u^{(\nu)}. \tag{38}$$

The $\alpha_u^{(\nu)}$ coefficients ($u=0$ to v; $\nu=1, N$) have to be calculated in terms of the potential coefficients $d_{2k}^{(\nu)}$.

From equations (29), (25) and (26), one gets

$$\begin{aligned}\alpha_0^{(\nu)}&=-B_0^{(\nu)}(0)\\ \alpha_u^{(\nu)}&=(-2)^u d_{2u}^{(\nu)}-\sum_{l=0}^{u}(-4)^{u-l}(2u-2l-1)!!B_{2u-2l}^{(\nu)}(l).\end{aligned} \tag{39}$$

The way of calculating the $B^{(\nu)}$ has been detailed in the preceding section. As previously done, we set $d_{2u}^{(1)}=d_{2u}$ and $d_{2u}^{(\nu)}=0$ for $\nu\geqslant 2$.

At the first order of the perturbation ($N=1$), $B^{(1)}=0$, $\alpha_0^{(1)}=0$, $\alpha_v^{(1)}=(-2)^v d_{2v}$ and one gets

$$E_v^{(1)}=b\sum_{u=1}^{u_1}2^u\binom{v}{u}d_{2u} \tag{40}$$

where $u_1=\min(S_1+1, v)$.

At the Nth order of the perturbation, one finds

$$E_v^{(N)}=b\sum_{u=0}^{u_N}\binom{v}{u}\sum_{l=0}^{u}(-)^{l+1}2^{2u-2l}(2u-2l-1)!!B_{2u-2l}^{(N)}(l) \tag{41}$$

where $u_N=\min(S_N+1, v)=\min(NS_1+1, v)$.

For the sake of computational simplification, it is convenient to perform the inversion of the sum $E_v^{(\nu)}$ (Jordan 1965). Thus, one gets the following expression ($\nu\geqslant 2$)

$$E_v^{(\nu)}=\sum_{u=1}^{v}(-)^{u+1}\binom{v}{u}E_{v-u}^{(\nu)}+(-)^v b\alpha_v^{(\nu)}. \tag{42}$$

Particularly, for $S_1 = 2$

$$E_0^{(1)} = 0, \qquad E_1^{(1)} = 2d_2 b, \qquad E_2^{(1)} = 4(d_2+d_4)b \qquad E_3^{(1)} = 2(3d_2+6d_4+4d_6)b$$
$$E_0^{(2)} = -(\tfrac{1}{2}d_2^2 + \tfrac{1}{3}d_4^2 + \tfrac{4}{15}d_6^2)b$$
$$E_1^{(2)} = E_0^{(2)} - (d_2^2 + 4d_2d_4 + 4d_4d_6 + 4d_6^2 + 4d_4^2)b \tag{43}$$
$$E_2^{(2)} = 2E_1^{(2)} - E_0^{(2)} - (8d_2d_4 + 18d_4^2 + 12d_2d_6 + 40d_6^2 + 48d_4d_6)b$$
$$E_3^{(2)} = 3E_2^{(2)} - 3E_1^{(2)} + E_0^{(2)} - \left(24d_2d_6 + \frac{68}{3}d_4^2 + 168d_4d_6 + \frac{664}{3}d_6^2\right)b.$$

It is worthwhile to compare expression (40) of the first-order energy with the usual expression

$$E^{(1)} = \sum_{u=1}^{v} 2^u \binom{v}{u} \sum_{t=u}^{S_1+1} C'_{2t} \frac{(2t)!}{t!(4b)^t} \binom{t}{u} \tag{44}$$

which is obtained when expanding the perturbation as a series $\Sigma_t C'_{2t} x^{2t}$ instead of a series $\Sigma_u u!/(2u)!\, d_{2u} H_{2u}(x)$. Obviously, formula (40) which involves only one summation is more compact than formula (44). This simplification is not a matter of algebraic manipulation, it is a telescopic effect: indeed, the two series are not equivalent unless the purely mathematical assumption $S_1 \to \infty$ holds. Furthermore, it is well known that the nth degree least-squares polynomial approximation to a function $f(x)$ over $(-\infty, +\infty)$, relevant to the weighting function $w(x) = e^{-x^2}$ is defined as a series of Hermite polynomials $H_r(x)$ with $r = 0$ to n (Hildebrand 1956). Thus the expansion of the perturbation on the Hermite polynomial basis constitutes a rearrangement of the x^{2t} which obviates the difficulties of convergence. This simplification still persists at the higher orders of the perturbation.

3.4. Eigenfunctions of the perturbed harmonic oscillator

Let us now consider the determination of the eigenfunctions of equation (13) in the form

$$\psi_v = \psi_v^{(0)} + \eta\psi_v^{(1)} + \eta^2\psi_v^{(2)} + \ldots + \eta^N\psi_v^{(N)} \tag{45}$$

with

$$\psi_v^{(\nu)} = \sum_{t=0}^{T_\nu+v} (2^t t!)^{1/2} a_t^{(\nu)}(v)\psi_t^{(0)} \qquad a_t^{(0)}(v) = \delta_{vt}(2^v v!)^{-1/2}.$$

Keeping in mind that the $\psi_t^{(0)}$ wavefunctions are solutions of the unperturbed eigenequation associated with the eigenvalues $E_t^{(0)} = 2b(t+\frac{1}{2})$, one can write, at each order N of the perturbation,

$$\sum_{t=0}^{T_N+v} 2b(v-t)(2^t t!)^{1/2} a_t^{(N)}(v)\psi_t^{(0)} = \sum_{\nu=1}^{N} (V^{(\nu)} - E_v^{(\nu)}) \sum_{t=0}^{T_{N-\nu}+v} (2^t t!)^{1/2} a_t^{(N-\nu)}(v)\psi_t^{(0)}. \tag{46}$$

From this expression and the expansion (14) of the $V^{(\nu)}$ on the Hermite polynomial basis together with the expression (17) of the $\psi_t^{(0)}$ functions, one finds

$$a_t^{(N)}(v) = \frac{1}{2b(t-v)} \sum_{\nu=1}^{N} \left(a_t^{(N-\nu)}(v)E_v^{(\nu)} - \sum_{s=0}^{T_{N-\nu}+v} \sum_{u=1}^{S_\nu+1} \frac{u!}{(2u)!} h(2u, s; t) d_{2u}^{(\nu)} a_s^{(N-\nu)}(v) \right) \tag{47}$$

where $T_N \geqslant 2S_\nu + T_{N-\nu} + 2$ i.e. $T_N = 2N(S_1+1)$; the $h(\ )$ are the coefficients involved in the product of two Hermite polynomials (equation (A1.3)) and analytical expressions of the energies $E_v^{(\nu)}$ in terms of the potential parameters d_{2k} have been determined in the preceding section.

This expression stands for $t \neq v$; for $t = v$, one can determine the $a_v^{(N)}(v)$ coefficients from orthonormalisation considerations. Indeed, we must impose

$$\sum_{\nu=0}^{N} \int_{-\infty}^{+\infty} \psi_v^{(\nu)*}(X)\psi_v^{(N-\nu)}(X)\,\mathrm{d}X = 0. \tag{48}$$

Since the unperturbed functions $\psi_v^{(0)}$ are orthonormalised, one gets the following expression

$$a_v^{(N)}(v) = -\frac{1}{2(2^v v!)^{1/2}} \sum_{\nu=1}^{N-1} \sum_t 2^t t!\, a_t^{(\nu)}(v) a_t^{(N-\nu)}(v). \tag{49}$$

As was previously done, we set $d_{2u}^{(1)} = d_{2u}$ and $d_{2u}^{(\nu)} \equiv 0$ for $\nu \geqslant 2$.

Particularly, for $N = 1$ and $v = 0, 1, 2, 3$, one gets the following expressions

$$\begin{aligned}
\psi_0 &= \psi_0^{(0)} - \frac{\sqrt{2}}{4b} d_2\psi_2^{(0)} - \frac{1}{b\sqrt{4!}} d_4\psi_4^{(0)} - \frac{4}{b\sqrt{6!}} d_6\psi_6^{(0)} \\
\psi_1 &= \psi_1^{(0)} - \frac{1}{b\sqrt{3!}}\left(\frac{3}{2}d_2 + 2d_4\right)\psi_3^{(0)} - \frac{1}{b\sqrt{5!}}(5d_4 + 6d_6)\psi_5^{(0)} - \frac{28}{b\sqrt{7!}} d_6\psi_7^{(0)} \\
\psi_2 &= \psi_2^{(0)} + \frac{\sqrt{2}}{4b} d_2\psi_0^{(0)} - \frac{\sqrt{3}}{2b}\left(d_2 + \frac{8}{3}d_4 + 2d_6\right)\psi_4^{(0)} - \frac{\sqrt{10}}{b}\left(\frac{1}{4}d_4 + \frac{3}{5}d_6\right)\psi_6^{(0)} - \frac{2\sqrt{35}}{15b} d_6\psi_8^{(0)} \\
\psi_3 &= \psi_3^{(0)} + \frac{\sqrt{6}}{4b}\left(d_2 + \frac{4}{3}d_4\right)\psi_1^{(0)} - \frac{\sqrt{5}}{b}\left(\frac{1}{2}d_2 + 2d_4 + 3d_6\right)\psi_5^{(0)} \\
&\quad - \frac{\sqrt{210}}{2b}\left(\frac{1}{6}d_4 + \frac{3}{5}d_6\right)\psi_7^{(0)} - \frac{2\sqrt{105}}{15b} d_6\psi_9^{(0)}.
\end{aligned} \tag{50}$$

4. Illustrative application

Let us now consider the resolution of the eigenequation (1) viewed as a particular case of the eigenequation (13). We set

$$\mathscr{V}(X) = b^2X^2 + (1-b^2)X^2 + (\lambda/g) - (\lambda/g)(1+gX^2)^{-1}. \tag{51}$$

The last terms of $\mathscr{V}(X)$ are expandable as convergent series of Hermite polynomials (see appendix 1).

Thus, one gets

$$\mathscr{V}(X) = b^2X^2 + \frac{\lambda}{g}(1-C_0) + \frac{1-b^2}{2b} + \left(\frac{1-b^2}{4b} - \frac{\lambda}{g}C_2\right)H_2(b^{1/2}X) - \frac{\lambda}{g}\sum_{k=2} C_{2k}H_{2k}(b^{1/2}X). \tag{52}$$

The resolution of the eigenequation (1) amounts to the resolution of the eigenequation

(13) with the following correspondence

$$\mathscr{E} = E_{jm} + (\lambda/g)(1-C_0) + (1-b^2)/2b, \qquad d_2 = -(2\lambda/gb)C_2 + (1-b^2)/2b^2,$$
$$d_{2k} = -\frac{\lambda}{gb}\frac{(2k)!}{k!}C_{2k}, \qquad k \geq 2, \qquad C_{2k} = \sum_{u=0}^{k} (-)^u (u!(2k-2u)!2^{2u})^{-1} I_{k-u}. \tag{53}$$

The basic integrals I_u have been previously calculated (Bessis and Bessis 1980):

$$I_u = \left(-\frac{b}{g}\right)^u I_0 - \sum_{s=0}^{u-1} \frac{(2s-1)!!}{2^s}\left(-\frac{b}{g}\right)^{u-s} \tag{54}$$

where I_0 is defined in terms of the complementary error function

$$I_0 = \sqrt{\pi}(b/g)^{1/2} \, e^{b/g} \operatorname{erfc}[(b/g)^{1/2}]. \tag{55}$$

One gets

$$d_2 = -(\lambda/gb)[I_0(\tfrac{1}{2}+b/g) - b/g] + (1-b^2)/2b^2 \tag{56}$$

and for $k \geq 2$

$$d_{2k} = -\frac{\lambda}{gb}\frac{1}{k!}\left[\left(-\frac{b}{g}\right)^k I_0 - \sum_{s=0}^{k-1}\frac{(2s-1)!!}{2^s}\left(-\frac{b}{g}\right)^{k-s}\right]$$
$$- \sum_{u=1}^{k} \frac{(2k)!(k-u)!}{k!2^{2u}u!(2k-2u)!} d_{2k-2u}. \tag{57}$$

Particularly

$$d_4 = -\frac{\lambda}{2gb}\left[I_0\left(\frac{b^2}{g^2}+3\frac{b}{g}+\frac{3}{4}\right) - \frac{b}{g}\left(\frac{b}{g}+\frac{5}{2}\right)\right]$$
$$d_6 = -\frac{\lambda}{6gb}\left[-I_0\left(\frac{b^3}{g^3}+\frac{15}{2}\frac{b^2}{g^2}+\frac{45}{4}\frac{b}{g}+\frac{15}{8}\right) + \frac{1}{g}\left(\frac{b^2}{g^2}+\frac{7b}{g}+\frac{33}{4}\right)\right]. \tag{58}$$

Once the d_{2k} coefficients have been calculated in terms of the potential parameters λ, g and b, the expressions of the energies, up to a given order N of the perturbation, directly follow from relation (42). Particularly, for $S_1 = 2$, explicit expressions are obtained, at the first and second order of the perturbation from relations (43). For the states $v = 0$ to 3, one finds again the expressions of the first-order energies already calculated in our previous paper (Bessis and Bessis 1980, equations (22) and (23)).

The expressions of the associated eigenfunctions, up to a given order N of the perturbation, follow from § 3.4. For the states $v = 0$ to 3, they are given explicitly in terms of the potential parameters by relations (50), (56) and (58).

5. Formulae for a perturbing potential function of any parity

Even though some of the formulae have been given in §§3 and 4 for the specific case of a perturbed harmonic potential function, even in X, the same technique is easily extendable to the case of any perturbing potential function which can be expanded in a series of Hermite polynomials $H_n(x)$ of both parities. Indeed, let us consider again the solution of eigenequation (13) with

$$V^{(\nu)}(b^{1/2}X) = \sum_{k=1}^{2S_\nu+2} C_k^{(\nu)} H_k(b^{1/2}X). \tag{59}$$

Using the same notations as in § 3, it is found that the expression (23) of the factorisation function $L(m)$ is still valid, and that the expressions (24) of the ladder function $K^{(N)}(x, m)$ and of the product $\Sigma_{\nu=1}^{N-1} K^{(\nu)}K^{(N-\nu)}$ still hold when setting

$$\begin{aligned} P_u^{(N)}(x) &= \sum_{t=0}^{t_M} A_t^{(N)}(u) H_t(x), \qquad t_M = 2(S_N - u) + 1 \\ Q_u^{(N)}(x) &= \sum_{t=0}^{t_M} B_t^{(N)}(u) H_t(x), \qquad t_M = 2(S_N - u + 1) \end{aligned} \tag{60}$$

and $S_N = NS_1$.

Relations (27)–(32) remain unchanged. The only change occurs in the expressions of the $A_t^{(N)}(u)$ and $B_t^{(N)}(u)$ coefficients. Their respective expressions (33) and (34) have to be simply generalised in the following way

$$A_t^{(N)}(u) = (-4)^u \frac{(t+2u)!!}{t!!} C_{t+2u+1}^{(N)} - \sum_{\sigma=0}^{u} (-4)^{u-\sigma} \frac{(2u-2\sigma+t)!!}{t!!} B_{t+2u-2\sigma+1}^{(N)}(\sigma) \tag{61}$$

$$B_t^{(N)}(u) = \sum_{s=0}^{u} \sum_{\sigma=0}^{u} (-)^{u+s+\sigma} \binom{u}{s} \binom{s}{u-\sigma} \sum_{\nu=1}^{N-1} \sum_{k=0}^{k_M} \sum_{q=0}^{q_M} h(k, q; t) A_k^{(\nu)}(s) A_q^{(N-\nu)}(\sigma) \tag{62}$$

with $k_M = 2(S_\nu - s) + 1$; $q_M = 2(S_{N-\nu} - \sigma) + 1$. Introducing the above definitions of the $A_t^{(N)}(u)$ and $B_t^{(N)}(u)$ coefficients, the expressions (37)–(42) are still valid and can be used to calculate the perturbed harmonic oscillator energies.

Proceeding to the calculation of the eigenfunctions, the same invariance of the formulae occurs for relations (45), (46) and (49): of course, since it was specialised to the even case, formula (47) has to be generalised by the following one

$$a_t^{(N)}(v) = \frac{1}{2b(t-v)} \sum_{\nu=1}^{N} \left(a_t^{(N-\nu)}(v) E_v^{(\nu)} - \sum_{s=0}^{T_{N-\nu}+v} \sum_{u=1}^{2S_\nu+2} h(u, s; t) C_u^{(\nu)} a_s^{(N-\nu)}(v) \right) \tag{63}$$

with $T_N = 2N(S_1 + 1)$.

6. Conclusion

Although the prime motivation of our investigation has been to solve analytically the Schrödinger equation with the potential $\mathscr{V}(X) = X^2 + \lambda X^2/(1 + gX^2)$, in fact, we have elaborated new techniques for handling efficiently the perturbed ladder operators method. In doing so, we have bypassed some difficulties of convergence and simplified the computational algorithm. This has been possible by using an Hermite working basis instead of the familiar x^k basis and by introducing the associated integral lowering operator $\hat{D}$. The finite difference solution in m of the factorisation condition and the embedding process (via the artificial parameter μ) have been greatly simplified by the consideration of binomial functions $\binom{m-\mu}{k}$. For all these reasons, it follows that analytical expressions of the eigenenergies and of the eigenfunctions can be obtained up to the Nth order of the perturbation without increasing intricacy. This last feature proves to be very useful when an elaborate perturbative resolution of the nuclear diatomic vibration–rotation wave equation is needed. This case is encountered, for instance, in the determination of the centrifugal distortion contributions to the rotational spectra of diatomics when the radial dependence of the fine structure interaction terms are taken into account (Bessis and Tergiman 1982).

One can conjecture that the procedure already outlined will be of great interest and efficiency when considering any wave equation which can be viewed as a 'perturbed factorisable' equation: very likely one has, for each kernel factorisable case under consideration, to work on the specific x-basis (Laguerre, Jacobi polynomials . . .), to introduce the associated integral lowering operator $\hat{D}$ and to choose the adequate finite-difference basis m-function. Results of this last investigation will be given elsewhere.

Appendix 1. Some properties of the Hermite polynomials

In the main text, we have used directly or indirectly the following properties of the Hermite polynomials.

A1.1. Definition

The Hermite polynomial of degree n is written

$$H_n(x)=\sum_{k=0}^{[n/2]}\frac{(-)^k n!}{k!(n-2k)!}(2x)^{n-2k}. \tag{A1.1}$$

Conversely

$$x^n=\frac{n!}{2^n}\sum_{t=0}^{[n/2]}\frac{1}{(n-2t)!t!}H_{n-2t}(x). \tag{A1.2}$$

A1.2. Product of two Hermite polynomials

$$H_u(x)H_s(x)=\sum_{t=0}^{u+s}h(u,s;t)H_t(x) \tag{A1.3}$$

where

$$h(u,s;t)=2^{g-t}u!\,s!/\Gamma(g+1-u)\Gamma(g+1-s)\Gamma(g-t+1)$$

$$2g=u+s+t.$$

In particular, $h(u,s;0)=\delta_{us}2^u u!$

A1.3. Step and multistep operators

$$\begin{aligned}&(\mathrm{d}/\mathrm{d}x-2x)H_n(x)=-H_{n+1}(x)\\&\mathrm{d}H_n(x)/\mathrm{d}x=2nH_{n-1}(x).\end{aligned} \tag{A1.4}$$

In order to solve equations (27) and (28) it is convenient to define, in the basis of the Hermite polynomials, an operator $\hat{D}$, right inverse of $(\mathrm{d}/\mathrm{d}x-2x)$, such that

$$(\mathrm{d}/\mathrm{d}x-2x)\hat{D}H_n(x)=H_n(x). \tag{A1.5}$$

Thus, $\hat{D}$ is defined, for $n \geqslant 1$, by

$$\hat{D}H_n(x) = -e^{x^2} \int_x^{+\infty} H_n(y)\, e^{-y^2}\, dy. \tag{A1.6}$$

It is easily found that, for $n \geqslant 1$, one gets

$$\hat{D}H_n(x) = -H_{n-1}(x) \qquad 2\hat{D}(d/dx)H_n(x) = -4nH_{n-2}(x)$$
$$[2\hat{D}(d/dx)]^t H_n(x) = (-4)^t[(n)!!/(n-2t)!!]H_{n-2t}(x) \tag{A1.7}$$
$$[2\hat{D}(d/dx)]^t \hat{D}H_n(x) = -(-4)^t[(n-1)!!/(n-2t-1)!!]H_{n-2t-1}(x).$$

A1.4. Expansion of a function $f(x)$ on the Hermite basis

A real function $f(x)$, defined for $-\infty < x < \infty$ and piecewise derivable, can be expanded as a convergent series on the basis of Hermite polynomials if the following integral (Dreszer 1975) exists:

$$\int_{-\infty}^{+\infty} |f(x)|^2 |x|\, e^{-x^2}\, dx.$$

$$f(x) = \sum_{n=0}^{\infty} C_n H_n(x) \tag{A1.8}$$

with

$$C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-x^2} H_n(x)\, dx. \tag{A1.9}$$

Let us consider the particular case $f(x) = (1+gx^2/b)^{-1}$. Using expansion (A1.1) for the $H_{2k}(x)$ polynomials, one gets

$$C_{2k} = \sum_{u=0}^{k} (-)^u \frac{1}{u!(2k-2u)!2^{2u}} I_{k-u} \tag{A1.10}$$

where, setting $x = b^{1/2}X$,

$$I_k = 2\left(\frac{b}{\pi}\right)^{1/2} \int_0^{\infty} e^{-bX^2} (b^{1/2}X)^{2k} \frac{dX}{1+gX^2}. \tag{A1.11}$$

Appendix 2. Calculation of $\Sigma_{\nu=1}^{N-1} K^{(\nu)} K^{(N-\nu)}$

From expression (18) of the ladder functions and using the relation (A1.3), one gets

$$\sum_{\nu=1}^{N-1} K^{(\nu)}(x,m) K^{(N-\nu)}(x,m)$$
$$= \sum_{\nu=1}^{N-1} \sum_{s=0}^{S_\nu} \sum_{t=0}^{S_{N-\nu}} \gamma_{2s+1}^{(\nu)}(m) \gamma_{2t+1}^{(N-\nu)}(m) H_{2s+1}(x) H_{2t+1}(x)$$
$$= \sum_{u=0}^{S_\nu + S_{N-\nu}+1} w_{2u}^{(N)}(m) H_{2u}(x) \tag{A2.1}$$

where

$$w_{2u}^{(N)}(m)=\sum_{\nu=1}^{N-1}\sum_{s=0}^{S_\nu}\sum_{t=0}^{S_{N-\nu}} h(2t+1,2s+1;2u)\gamma_{2s+1}^{(\nu)}(m)\gamma_{2t+1}^{(N-\nu)}(m)$$

and $h(\ )$ is the coupling coefficient of two Hermite polynomials. When comparing the values of the upper bounds in (A2.1) and in (20), it is found that the following necessary condition must hold

$$S_N=S_\nu+S_{N-\nu} \tag{A2.2}$$

and consequently, $S_N=NS_1$.

At the first order $N=1$ of the perturbation, since $w^{(1)}(m)\equiv 0$, the starting function $\gamma_{2s_1+1}^{(1)}(m)$ of the recursive process is the solution of the finite difference equation $\Delta\gamma_{2s_1+1}^{(1)}(m)=0$ (equation (19)). It is easily inferred that $\gamma_{2t+1}^{(1)}(m)$ is a polynomial in m of degree (S_1-t). Then, the product $\gamma_{2t+1}^{(1)}(m)\gamma_{2s+1}^{(1)}(m)$ is a polynomial of degree $(2S_1-t-s)$. From (A2.1), it is easily seen that a given $w_{2u}^{(2)}(m)$ is generated from the products $\gamma_{2t+1}^{(1)}\gamma_{2s+1}^{(1)}$ such that $s+t+1>u$. Thus, $w_{2u}^{(2)}(m)$ is a polynomial in m of degree $(2S_1-u+1)=(S_2-u+1)$ and finally, both $\gamma_{2t+1}^{(N)}(m)$ and $w_{2t+2}^{(N)}(m)$ are polynomials in m of degree (S_N-t).

In the framework of the finite-difference calculus, the expressions (24) can be interpreted as Newton's backward formulae for $K^{(N)}(x,m)$ and for $\Sigma_\nu K^{(\nu)}K^{(N-\nu)}$, respectively with

$$\begin{aligned}P_u^{(N)}(x)&=[\Delta^u K^{(N)}(x,m)]_{m=\mu-u}\\ Q_u^{(N)}(x)&=\sum_{\nu=1}^{N-1}[\Delta^u K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)]_{m=\mu-u}.\end{aligned} \tag{A2.3}$$

Using a discrete Leibniz rule, i.e. the expression of the uth finite difference Δ^u of a product, one obtains (Jordan 1965)

$$Q_u^{(N)}(x)=\sum_{\nu=1}^{N-1}\sum_{s=0}^{u}\sum_{t=0}^{u}(-)^{u+s+t}\binom{u}{s}\binom{s}{u-t}P_s^{(\nu)}(x)P_t^{(N-\nu)}(x). \tag{A2.4}$$

Few simple manipulations are necessary in order to obtain the expression (34) of the $B_{2t}^{(N)}$ coefficients of $Q_u^{(N)}(x)$ in terms of the coefficients $A_{2u+1}^{(\nu)}(t)$ of the $P_t^{(\nu)}(x)$. The expression (25) is substituted for the $P_t^{(\nu)}$ into relation (A2.4). Then using relation (A1.3), one obtains an expansion of $Q_u^{(N)}(x)$ on the basis of the Hermite polynomials. Since this expansion of $Q_u^{(N)}$ has to be identical to expression (26), one obtains the required relation (34).

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